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The general solution of the homogeneous complex Monge–Ampère equation in a space of arbitrary dimension

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Abstract. A general solution to the complex Monge–Ampére equation in a space of arbitrary dimensions is constructed.

1. Introduction

The homogeneous complex Monge–Ampére equation (HCM–A) in n-dimensional space has the form:

$$\det \begin{vmatrix} \frac{\partial^2\phi}{\partial y_1 \partial \bar{y}_1} & \cdots & \frac{\partial^2\phi}{\partial y_1 \partial \bar{y}_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2\phi}{\partial y_n \partial \bar{y}_1} & \cdots & \frac{\partial^2\phi}{\partial y_n \partial \bar{y}_n} \end{vmatrix} = 0. \tag{1}$$

Its real form, which arises from (1) under the assumption that the solution depends only upon n arguments, $x_i = y_i + \bar{y}_i$, has been found before by different methods [1, 2], but to the best of our knowledge the general solution of the HCM–A equation (1) is still wanting.

The aim of the present paper is to fill this gap by using the method of our previous paper [2] (the solution of the real HM–A equation in a space of arbitrary dimension) to obtain and present the general exact solution of the complex version of this equation in implicit form.

In order to understand the nature of our solution all that is necessary is a knowledge of the rules of differentiation of implicit functions. The Monge–Ampére equation is an equation of second order and thus its general solution (in the Cauchy–Kovalevsky sense) must be dependent upon two arbitrary functions each of $(2n-1)$-independent arguments. A solution of exactly this form is presented in the theorem found in the next section. All questions about existence of singular solutions (shock waves for instance) lie outside the framework of this paper.

2. General case of arbitrary n

Theorem. *Let the set of functions $\psi^\alpha(y;\bar{y})$ be determined implicitly by the following set of equations, the number of which coincides with the number of functions ψ:*

$$\Theta_{\psi^\beta}(\psi; y) = -\bar{\Theta}_{\psi^\beta}(\psi; \bar{y}) \tag{2}$$

where Θ, $\bar{\Theta}$ *are arbitrary differentiable functions of their* $(2n-1)$ *arguments* (y_i, ψ^α), $(\bar{y}_i, \psi^\alpha)$, $1 \leqslant i \leqslant n$, $1 \leqslant \alpha \leqslant (n-1)$ *and the subscript denotes differentiation with respect to* ψ^β. *Then, selfconsistent derivatives of the function* ϕ *satisfying the HCM–A equation in n dimensions are determined with the help of the formulae*

$$\phi_y = \Theta_y \qquad \phi_{\bar{y}} = \bar{\Theta}_{\bar{y}}. \tag{3}$$

Equation (2) implicitly defines $(n-1)$ functions ψ^α as functions of $2n$ arguments $y, \bar{y}$.

Now we would like to prove that derivatives of the function ϕ determined with the help of formulae (2) are selfconsistent in the sense of equality of the second mixed partial derivatives with respect to all independent arguments involved.

Let us first check the conditions of selfconsistency of the second mixed derivatives with the same (barred, unbarred) indices. We consequently have (for two arbitrary coordinates (y_1, y_2))

$$(\phi_{y_1})_{y_2} = \Theta_{y_1,y_2} + \sum \Theta_{y_1,\psi^\nu}\psi^\nu_{y_2} = \Theta_{y_1,y_2} + \sum \Theta_{y_1,\psi}(\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}\Theta_{y_2,\psi}.$$

In writing the last equality we have used the explicit expressions for derivatives of the functions ψ, and the vector component superscript on ψ has been suppressed in the final expression. These derivatives follow directly from equation (2) differentiated with respect to the arguments $y(\bar{y})$:

$$\psi_y = -(\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}\Theta_{\psi,y}$$
$$\psi_{\bar{y}} = -(\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}\bar{\Theta}_{\psi,\bar{y}}.$$

The matrix $(\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}$ is obviously symmetric, so the last expression is symmetric with respect to permutation of the indices (1, 2). Thus second mixed derivatives with indices of the same kind are selfconsistent.

Now let us calculate the mixed derivatives with indices of different kinds:

$$(\phi_{y_i})_{\bar{y}_k} = -\sum \Theta_{y_i,\psi^\nu}\psi^\nu_{\bar{y}_k} = \sum \Theta_{y_i,\psi}(\Theta_{\psi,\psi} + \bar{\Theta}_{\psi,\psi})^{-1}\bar{\Theta}_{\bar{y}_k,\psi}. \tag{4}$$

The result of calculation in the opposite order gives exactly the same result also as a corollary of the symmetry of the same matrix.

Thus we have proved that derivatives of the ϕ function are determined in a selfconsistent manner and can be reconstructed by integration of the differential one-form $\mathrm{d}\phi$ as a line integral;

$$\phi = \sum_i \left(\int \mathrm{d}y_i \Theta_{y_i} + \int \mathrm{d}\bar{y}_i \bar{\Theta}_{\bar{y}_i} \right).$$

It is seen from the explicit expression for second derivatives of the ϕ function $\phi_{y_i,\bar{y}_k}$, calculated above, that the condition of linear dependence between its rows is equivalent to the equality

$$\sum_{i=1}^{n} \mathrm{d}_i \Theta_{y_i,\psi} = 0.$$

We assume that the determinant of one among the n $(n-1)\times(n-1)$ matrices $Q_{\psi^\alpha,y}$ is different from zero (otherwise there exists a functional dependence among the $(n-1)$ functions ψ^α); then the last equality is equivalent to a linear system of algebraic equations:

$$Q_{\psi,y_n} + \sum_{\nu=1}^{n-1} \mathrm{d}_\nu Q_{\psi,y_\nu} = 0$$

from which all coefficients of the linear dependence are uniquely determined.

Thus we have proved that linear dependence occurs among n rows of the matrix $\phi_{y_i,\bar{y}_k}$, so its determinant is equal to zero and the function ϕ is determined by the conditions of the theorem to satisfy the HCM–A equation in the n-dimensional case.

We emphasize once more that this solution is determined by two arbitrary functions Θ, $\bar{\Theta}$ each depending on $(2n-1)$-independent arguments. In this sense we say that the solution constructed is the general one in the Cauchy–Kovalevsky sense.

2.1. Example

It is evident that a large class of explicit solutions to (1) can be constructed as follows. Let

$$\phi = \sum_{k=1}^{k=n-1} f^k(y)\bar{f}^k(\bar{y}) \tag{5}$$

where the $(n-1)$ arbitrary functions $f^k(y)$ depend only upon the variables y_j, and the functions $\bar{f}^k(\bar{y})$ similarly depend only upon the barred variables. Then it is straightforward to verify that this is a solution to (1). How does this fit with our construction? If one takes

$$\Theta(\psi^k, y) = \sum_{k=1}^{k=n-1} (\psi^k - f^k(y))^2 \qquad \bar{\Theta}(\psi^k, y) = \sum_{k=1}^{k=n-1} (\psi^k - \bar{f}^k(y))^2$$

and follows the procedure described here then solution (5) is recovered.

3. Outlook

The main result of this paper is presented in the theorem in section 2, giving the possibility of finding the general solution of the HCM–A equation (1) in implicit form. We especially emphasize that we cannot say that we have found all solutions of this equation but only those in which the number of arbitrary functions and their functional dependence is sufficient to resolve the statement of the problem of solution of the HM–A equation in terms of initial data of the Cauchy–Kovalevsky type. This solution is the most nondegenerate and excludes solutions of the shock-wave type. Similar results have been found for the complex Bateman equation [3].

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